International Journal of Philosophy
2022; 10(2): 90-95
http://www.sciencepublishinggroup.com/j/ijp
doi: 10.11648/j.ijp.20221002.16
ISSN: 2330-7439 (Print); ISSN: 2330-7455 (Online)

SciencePG
Science Publishing Group

The Logical Study of Non-Well-Founded Set and Circulation Phenomenon

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To cite this article:
Shi Jing. The Logical Study of Non-Well-Founded Set and Circulation Phenomenon. *International Journal of Philosophy*. Vol. 10, No. 2, 2022, pp. 90-95. doi: 10.11648/j.ijp.20221002.16

Received: June 12, 2022; **Accepted**: June 21, 2022; **Published**: June 29, 2022

Abstract: In recent years, since a great deal of circular phenomenon, there has been a furry of interest in them. To explain various circular phenomenon, the study of set theory extended well -founded sets to non-well-founded set. Based on this basis, the paper discusses the logical theoretical basis of circular phenomena. Non-well-founded set theory ZFA allows primitive existence. Primitive is an object that has no elements and is not a class in itself. It is based on the set theory ZFC after the axiom of foundation FA is removed, and the anti-basic axiom AFA is added to ZFC. ZFC here refers to ZF set theory with axiom of choice. According to axiomatic set theory, for ZFC's regular axioms, the set in its universe is a well set. If the regular axiom is removed, and the infinite decline is allowed to belong to the relational chain, then the non-well-founded set can be introduced. Firstly, this paper introduces the basic concept of non-well-founded set, the foundation axiom and the anti-founded axioms. Secondly, we dicusses the limit of the foundation axiom. Thirdly, we exhibit the history and present situation of the research on non-well-founded sets are briefly reviewed. Finally, the applications of non-well-founded sets in philosophy, linguistics, computer science, economics and many other fields is discussed. Because non-well-founded set theory will provide a better tool for dealing with circular phenomena naturally, it can be argued that circle is not vicious.

Keywords: Non-Well-Founded Set, Circulation, Set Theory, Mathematical Logic

1. Introduction

In recent 30 years, the phenomenon of circulation has attracted the attention of researchers in the fields of artificial intelligence, computer science, cognitive science, linguistics and philosophy. For example, computer scientists who are interested in the concepts of process and flow have started to make mathematical explanations for circulation, mathematical linguists are working hard to explain the characteristic structure, philosophers are studying the theories of truth and reference, and artificial intelligence workers are studying the circulation phenomenon in terminology.

At the beginning of last century, the emergence of Russell's paradox and several other paradoxes of set theory shocked many famous mathematicians. In order to eliminate the paradox, the scholars of set theory systematically sorted out Cantor's theories and methods with the help of axiomatic methods, and established a variety of rigorous set theory systems. ZF set theory, which is widely used today, is one of the most famous ones. Intuitive sets are indistinguishable from classes, and one property determines a class. ZF set theory can't promise that all these classes are sets. It uses the axiom of foundation FA to ensure that the promised sets are all well-founded. In order to meet the needs of its own theoretical development and other fields to deepen the understanding of cyclic phenomena, the study of set theory extends from well-founded sets to non-well-founded sets, which is also called superset. Firstly, this paper introduces the basic concept of non-well-founded set. Then, the history and present situation of the research on non-well-founded sets are briefly reviewed. Finally, the applications of non-well-founded sets in philosophy, linguistics, computer science, economics and many other fields is discussed.

2. Non-Well-Founded Set

In ZFC axiomatic system, the axiom of regularity is to limit the domain of set theory to well-founded sets. Using the axiom of regularity, the following conclusions can be proved.

Theorem 1.1 For any set x, all $x \notin x$ holds.

Proof: Suppose there exists a set x such that $x \in x$. Obviously, for the set $\{x\} \neq \varnothing$, according to the axiom of regularity, $x \cap \{x\} = \varnothing$, but since $x \in x$, so $x \cap \{x\} = \{x\} \neq \varnothing$, contradicting.

Theorem 1.2 There is no set sequence $x_0, x_1, \ldots, x_n, \ldots$, such that $\ldots x_n \in x_{n-1} \in \ldots \in x_1 \in x_0$.

Proof: Suppose there exists a sequence of sets $x_0, x_1, \ldots, x_n, \ldots$, such that $\ldots x_n \in x_{n-1} \in \ldots \in x_1 \in x_0$. Let the set $S = \{x_0, x_1, \ldots, x_n, \ldots\}$. Obviously, for any natural number i, $x_i \in S$. According to the axiom of regularity, S has a $\in$-relation to a minimal element x_m, but $x_{m+1} \in x_m$, which contradicts the minimality of x_m.

Definition 1.1 If the set S has elements $x_0, x_1, \ldots, x_n, \ldots$, such that $\ldots \in x_n \in x_{n-1} \in \ldots \in x_1 \in x_0$, then $x_0, x_1, \ldots, x_n, \ldots$ is called an infinitely descending $\in$-chain of S.

Theorem 1.2 If $S \neq \varnothing$, and there is no an infinitely descending $\in$-chain in S, then there is a -minimum element in S.

Proof: Suppose there is no $\in$-minimum element in S. Then for every $x \in S$ there is $y \in S \cap x$. For any $x_0 \in S$, then there is an infinitely descending $\in$-chain$\ldots \in x_2 \in x_1 \in x_0$.

(1) $x \in x$;
(2) $x \in y \wedge y \in x$;
(3) $x \in y \wedge y \in z \wedge z \in x$;
(4) $x_0 \in x_1 \in x_2 \ldots \in x_n \in x_0$;
(5) $\ldots x_{n+1} \in x_n \in x_{n-1} \ldots \in x_1 \in x_0$.

Let property $\phi(x)$ denote "there exists an infinitely descending $\in$-chain starting from x". Combine this property with negation to obtain the property "$\neg\phi(x)$", and all the properties mentioned above satisfy $\neg\phi(x)$. The axiom of regularity holds if there is no infinitely descending $\in$-chain for any non-empty set. Therefore, the axiom of regularity is equivalent to no infinite-descending chain for any non-empty set. Therefore, in the ZFC axiom system, the axiom of regularity exclude the sets $x, y, z, x_0, \ldots, x_n, \ldots$, that satisfy the following conditions:

(1) $x \in x$;
(2) $x \in y \wedge y \in x$;
(3) $x \in y \wedge y \in z \wedge z \in x$;
(4) $x_0 \in x_1 \in x_2 \ldots \in x_n \in x_0$;
(5) $\ldots x_{n+1} \in x_n \in x_{n-1} \ldots \in x_1 \in x_0$.

A set that satisfies one of the above conditions is called a peculiar sets, also known as an non-well-founded set [1]. Therefore, a set x is non-well-founded if it satisfies $\phi(x)$. The axiom of regularity presupposes that all sets are well-founded, so non-well-founded sets are not the research object of *ZFC*, that is, sets in any domain of *ZFC* (such as *ZFC*'s basic model *M*, composable model *L*, and Boolean model) all are well-founded set.

The combination of a property $\phi(x)$ satisfied by an non-well-founded set and its negation gives rise to Russell's paradox. Consider class $X = \{x: \neg\phi(x)\}$. In set theory *ZFC*, all sets satisfy $\neg\phi(x)$, so X is equal to the universe V of *ZFC*, that is, the class composed of all sets, so it is not a set, but a proper class.

The fundamental axioms make sense because in an aggregated hierarchy of sets, it can be shown that a set x is well-founded if and only if it belongs to some hierarchy V_α. This means that all sets in the aggregation hierarchy are well-founded sets. However, the union $\bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$ of all aggregates is equal to the class V composed of all sets, which is the domain of set theory *ZFC*. Therefore, in the axiomatic set theory *ZFC*, it is determined that all sets are well-founded as an axiom, which is an important property of sets in the universe of discourse. When we study the model of set theory, the axiom of regularity is very important.

The original reason for the creation of axiomatic set theory was to study the basic problems of mathematics, and the development of set theory has always been to study the foundation of mathematics. Mathematical concepts concerned in set theory, such as: natural numbers, integers, rational numbers, real numbers, cardinal and ordinal numbers, functions, etc., their definitions and the proof of their properties do not need to use the axiom of regularity, so it does not matter whether there is a non-well-founded set will appear it is not important. Therefore, the introduction of non-well-founded sets does not harm the study of mathematical foundations.

Furthermore, since the non-well-founded set itself involves the phenomenon of cycles, it can be used to simulate cycles. Therefore, it can be widely used in many fields to solve some practical problems and has certain many value. For example: the framework and model of modal logic are directed graphs, and there are cycles in these directed graphs. Since non-well-founded sets can handle graphs with circular nodes, they can be used to study modal logic. In addition, non-well-founded sets have a wide range of applications in linguistics, economics, philosophy, and theoretical computer science. For example, many paradoxes in philosophy are circular phenomena.

3. Well-Founded Set and Non-Well-Founded Set

In the real world, circular phenomena can be seen everywhere. For example: the phenomenon of blood circulation in the human body; the time cycle of 365 days a year and 24 hours a day; the cycle phenomenon of climate change in the four seasons of spring, summer, autumn and winter; the circulation phenomenon of the four links of production, exchange, distribution and consumption in macroeconomic operation is orderly. etc. The circular phenomenon involves itself directly or indirectly, which conflicts with the axiom of regularity *FA* in the *ZFC* axiom set theory, and cannot be modeled by the classical *ZFC* set theory. Non-well-founded sets are also called supersets or peculiar sets. They study sets with cyclic properties, such as sets satisfying the properties $x \in x$ or $x \in x_n \in \ldots \in x_2 \in x_1 \in x$. Using non-well-founded sets, models can be constructed for cyclic phenomena. In the 1980s, the study of non-wel-founded sets has made significant progress, and it has played an important role in many fields such as philosophy, economics, modal logic, situational semantics, and theoretical computer science.

There are a lot of circulation problems in philosophy, which involve "self" to some extent. For example, Descartes believes that the premise of people's undeniable thinking is that we are thinking, that is, people can doubt everything that can be doubted, but only they can't doubt their own thinking, because doubts about their own thoughts and behaviors also require thinking, and this thinking activity also involves doubts. It is the cycle of thinking that helped him make his famous conclusion: I think, therefore I am.

Is this argument involving reflexivity reasonable? If we insist on the axiom of regularity of set theory, then reflexivity is unreasonable. Because any well-founded set is not reflexive, belonging to a relationship is not a reflexive relationship, and the unfolding of the reflexive relationship is an infinite descending chain. But if we think that there can be a self-returning relationship, then the thinking of doubt itself can be doubted. Therefore, Descartes' argument seems to require a strong assumption: all sets are well-founded or all relationships are well-founded. Regarding this problem, if we consider using non-well-founded sets, it can help us re-understand Descartes' thesis.

3.1. Circulation in Philosophy

The most striking circular phenomena in philosophy appear in paradoxes, including some logical and semantic paradoxes, such as: the paradox of lies, Russell's paradox, Conway's paradox, and the paradox of reference. All these paradoxes can be described by classes that are not n-cycle classes (n is an arbitrary natural number), or even all non-cyclic classes, and the resulting sets are non-well-founded sets. The 0-cycle class refers to a class with self-attributes. For a non-zero natural number n, a class X is cycled n times only if there are n classes $X_1, X_2, \ldots, X_n$ (not necessarily all the same), such that $X \in X_n \in \ldots \in X_1 \in X$ is established [2]. For a certain natural number n, if a class X is cyclic n times, then X is called cyclic, and the resulting set is an non-well-founded set. The combination of cycle and negation will produce a paradox, and the set in Russell's paradox is the non-well-founded set generated by the combination of 0-cycle class and negation. Therefore, to exclude this kind of paradox, it is necessary to exclude all non-n-cycle classes, and even the non-well-founded sets generated by all non-cycle classes. Therefore, we need to study models that make the individual domain contain other models in an non-well-founded way, especially self-reflexive models, that is, models that are themselves elements in their individual domain, and use this to explore the semantic paradox of self-referentiality. The purpose here is not to solve the paradox, but to construct a framework for explaining the occurrence of the paradox [3].

Conway paradox also leads to cycle, which has many forms: from "unfaithful spouse" to "mud child" to "participant" and so on. In addition to the study of paradox, non-well-founded set theory has been used by philosophers in recent years to study truth and reference theory. Using non-well-founded sets to simulate various cycles is an important new mathematical tool, which has a wider application prospect.

In 1969, D. Lewis first put forward the concept of common knowledge [4]. Later, in 1981, H. Clark and C. Marshall popularized Lewis' research results [5]. For an arbitrary proposition p, if every member in this group knows p, and every member knows that every member knows p, in addition, every member knows that every member knows that every member knows p, …, etc., then this proposition p is called the common knowledge of this group. The difference between one group and another is that the two groups have different common knowledge. Let a and b form a group, and both a and b know proposition p. At this time, p is the knowledge of a, and p is also the knowledge of b, but p is not the common knowledge of a and b; If a knows that b knows p, on the other hand, b also knows that A knows p, and both parties know that the other party knows that they know p, ……, then p is the common knowledge of a and b. There is a circulation phenomenon here. If you use conventional methods to construct models, you must form a set with itself as an element. If the tools provided by *AFA* are adopted, it is easier to construct a strict model and study the properties of the model deeply.

In addition to this, things in connotative phenomena are also cyclical in nature. Let's take belief as an example to illustrate this point. Suppose a first-order structure M is used to simulate possible worlds, and the proposition p is a set of possible worlds, that is, the set of possible worlds that makes p true. For subject a and proposition p, we can express belief as a binary relation $B_a\, p$ between subject a and proposition p. If $M \vDash B_a\, p$, the belief that $M \in p$ implies a is true in M. We can give a brief proof. If $M \vDash B_a p$ and the belief of a is true in M, then M is an non-well-founded structure. The first-order structure must "contain" what is being talked about, that is, if $M \vDash B_a p$, then p belongs to $TC(M)$. If a has a true belief p in M, then $M \in p$. At last, we can get $M \in p \in TC(M)$, $p \in \ldots \in M \in p$, so both M and p are non-well-founded.

In the field of game theory and economics, economists use game theory to simulate people's decision-making behavior in uncertain situations. Let I be the set of agents and W be the set of cognitive possible worlds. For each world $w \in W$, w is associated with a certain state $S(w)$ of possible worlds, including the payoff function of game theory and the object $t_i(w)$. In the world w, $t_i(w)$ can simulate the belief or probability of each participant $i \in I$. There is an implicit assumption in game theory that players are assumed to know the information structures of other players. For this assumption to be an explicit component of the model, there must be cycles in the world w, so Heifetz used non-well-founded sets as a tool to model game theory [6].

3.2. Circulation Phenomenon in Modal Logic

One of the important application fields of non-well-founded sets in logic is modal logic. Here I will simply explain how to use non-well-founded sets to study modal logic. For the study of relational structure, modal language is a language with strong expressive function. A relational structure is a non-empty set and a binary relation on this set. In fact, relational structure is a directed graph in mathematical graph theory. For a finite directed tree graph, a relationship of mathematical isomorphism can be established between a natural number n and the belonging relationship on this set. For any well-founded graph, according to the Mostowski-collapse theorem, there is a unique transitive model isomorphic to it.

So, what about the non-well-founded graph? If we assume the existence of non-well-founded sets, we can also associate non-well-founded graphs with non-well-founded sets. For example, there is a single point graph "↻" with a reflexive relation. It uses a set of its own single points to correspond to it. According to this connection, we can use modal language to talk about sets. In this way, from the perspective of model theory, modal logic can be studied under the new semantics of set theory.

In 1988, Peter Aczel first expressed the relationship between infinite modal logic and non-well-founded sets. Then L. Moss and others found that modal logic can be developed by the canonical Kripke structure with the anti-basic axiom. In 1989, J. Barwise. defined the fixed point model in the context of non-well-founded set theory. In 1993, L. Lismont used the neighborhood semantics to analyze the infinite iterative method in multimodal cognitive logic. In 1995, he used infinite iterations and loops (or fixed points) to define the coexistence of public knowledge, respectively [7]. In 1999, T. Tsujishita interpreted the epistemic modal formulation as a universal modal world in Aczel's non-well-founded universe. This approach eliminates the limitation of interpreting cognitive formulas in the modal world of a well-founded universe. After that, L. Mosszai proved that the non-well-founded set semantics of group declaration logic and the Kripke model semantics are equivalent. Afterwards, Lurey constructed the non-well-founded set theory and established the relationship between the "non-well-founded set on the allowable set" and the fragment L_A of the modal language L_∞.

In 1999, A. Baltag proved that any non-welll-founded set can be described by an infinite modal language formula. If it is limited to the modal language constructed by the set of finite propositional variables, this description is no longer valid. However, a non-well-founded set can be characterized by a certain formula of modal language constructed by using finite propositional argument sets. The necessary and sufficient condition is that the set is well-founded and its transitive closure is finite. [8] Therefore, the description of the set by modal formula depends on the modal language used. In particular, if a set of finite transitive closures is not well-founded, it can only be described by the formula of infinite modal language. However, it can also be proved by modal-calculus that a non-well-founded set can be characterized by a certain formula of modal μ-calculus if and only if its transitive closure is finite. In 2004, L. Alberucci and V. Salipante used the maximum fixed point rule of modalμ-calculus to prove that any non-well-founded set with finite transitive closure can be characterized by a certain formula of modal language constructed by finite propositional argument sets. This generalizes the results of Batag's proof and establishes a new connection between automata theory and non-well-founded sets [9].

3.3. Circulation Phenomenon in Linguistics

In linguistics, one of the application fields of non-well-founded set and anti-founded axiom AFA is situation semantics. Situation semantics is a model theory method to study the semantics of natural language. A situation can be regarded as a part of the world composed of facts. Every fact consists of a relation, an object sequence with this relation and a polarity, which expresses whether this object sequence has or does not have this relation according to the polarity. As situations themselves are objects, they can also be part of facts. Therefore, a situation can become a component of the facts in a situation, and it will naturally produce circular situations, which contain some facts about themselves. The way to deal with the situation is to express the situation as a set of facts and the facts as a triple (R, a,). R is the relation, and a is the sequence with relation R, and σ is the polarity 0 or 1 [10].

In 1983, J. Barwise and J. Perry didn't use non-well-founded sets in their works [11], but in 1987, J. Barwise and Edgemendy used the concepts of circular situation and circular proposition to discuss the liar paradox, and expressed this abstract object with non-well-founded sets. Situation semantics takes non-well-founded set theory as a meta-theory to describe situations, and it also uses non-well-founded set theory as a tool to describe whether things in different time and space have certain properties and whether things have certain relationships. Scholars such as Barwise have found that because of the paradox of set theory, if the situation is described by standard set theory, the axioms established in set theory will have many difficulties in explaining the situation. One of the problems is that it conflicts with the axiom of "set cannot belong to itself" in the model of set theory. Later, they regarded the situation as a basic entity, analyzed it mathematically, and then created the meta-theory of situational semantics. However, in daily communication, the situation of conversation can be completely related to self. So in 1990, Aczel used non-well-founded set theory as a tool to study structural objects, and put forward the research ideas of situation theory. The research can range from structural object theory to structural proposition theory, and then to mathematics. Therefore, we can effectively solve the phenomenon of circulation or self-reference by describing the circular situation with the model of non-well-founded set theory.

In linguistics, non-well-founded sets can also be used for circular phenomena in anaphora. In addition, in theoretical computer science, labeled-transition-system is a common methodl for studying operational semantics. In the calculation model, the typical structure is the labeled-transition-system. There are many circulation phenomena in this system. It can execute the same program as many times as it wants, or it can execute some programs back and forth between different computing states. For example, the following is an uncertain labeled-transition-system:

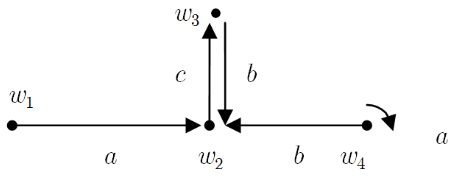

Figure 1. *An uncertain labeled-transition-system.*

Here, a is an indeterminate program. At state w_4, if program a is run a finite number of times, state w_4 can be reached. Then a loop is involved on state w_4. And there is also some kind of cycle between w_2 and w_3. If the program is run on w_3, the calculation state can only appear back and forth between w_2 and w_3.

Non-well-founded set theory is closely related to the labeled-transition-system. In 1996, L. Lazic and A. Roscoe constructed canonical labeled-transition-system and Associated mappings pedigree, and gave the strong extension theory of transformation system. Any two points with equivalent behaviors can be identified [12].

The semantics of terminal algebra based on non-well-founded set theory is a mathematical theory to express the semantics of structural operations. In addition, domain theory and referential semantics are established on the basis of non-well-founded sets. These theories are based on the interaction between the principle of induction and the non-standard "co-induction principle". The co-inductive properties of co-algebras are studied in the categories of classification layers and non-well-founded sets, partial orders and metric spaces, and it is proved that the basic category concept of final co-algebras is the basis of the principle of co-induction. In addition, in 1999, Lenisa investigate the relation between the set-theoretical description of coinduction based on Tarski Fixpoint Theorem, and the categorical description of coinduction based on coalgebras. Moreover, they investigate the connection between these and the equivalences induced by T-coiterative functions. These are morphisms into final coalgebras, satisfying the T-coiteration scheme, which is a generalization of the corecursion scheme. It show how to describe coalgebraic F-bisimulations as set-theoretical ones. [13]

In 2000, C. Piazza and A. Policriti put forward a process of judging the semantic graph of non-well-founded formula in order to judge the finiteness and satisfiability of non-quantified formula with "weak power set". This result can be applied to many non-well-founded set theories. The program defined by them can be used to determine the formula class of set theory without basic axioms [14]. In 2006, Berg described the phenomenon of cycle and non-downtime with a bad base tree, which can be used for semantic research of process theory and co-inductive type programming languages, and established a model for bad base sets and non-downtime processes or infinite data structures. And proved the stability of the category of labeled ill-founded trees in various Topos theoretical constructions [15]. Non-well-founded trees are used in mathematics and computer science, for modelling non-well-founded sets, as well as non-terminating processes or infinite data structures. Categorically, they arise as final coalgebras for polynomial endofunctors. These are then used to prove stability of such categories with M-types under various topos-theoretic constructions.

4. Conclusion

Acyclic non-well-founded sets can be viewed as cyclic classes with infinite cyclic nodes. The combination of circularity and negation will produce paradoxes. For example, Russell's paradox is the paradox of all classes of non-zero cyclic classes, and all classes of non-n cyclic classes, even all classes of non cyclic classes, also form paradoxes. The axiom of foundation FA requires all sets to be well-founded, and directly excludes circular classes and all non-well-founded sets. The establishment of non well founded set theory will not only break the rule of classical set theory, but also fundamentally innovate and expand people's understanding of sets. With the continuous study of set theory, it will also play an important role in promoting the understanding of other theoretical concepts, and it will also play an important role in more fields [16].

In addition to set theory, cycles also appear in many fields, such as philosophy, linguistics, computer science, economics and mathematics, which indicates that non well based set theory has a broad application prospect. Circular phenomena can be seen at any time in the physical, physiological and psychological world around us. Circularity is also very important in the design and research of engineering technology, especially in the design of computer system. The concept of state is used in the design of hardware. The designed system must be able to return to a given state in a normal way. People's understanding of the world is cyclical, and the significance of this cyclicality can be illustrated by the paradox of conway. Besides the paradox of conway, there are lie paradox, reference paradox, advanced game paradox and Russell paradox. Barwise used non-well-founded sets to create the "anti-basic model theory", he proved the Liar's Theorem in the non-well-founded set theory, and then compared with the existing solutions to the Liar's paradox. In the end he concluded that context sensitivity makes sense for explaining the intuitive reasoning behind the Liar's paradox [17]. After analyzing these paradoxes, it is found that they all involve some kind of interaction between circularity and negation, and circle is not vicious. Non-well-founded sets will provide tools to deal with these circular phenomena naturally.

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